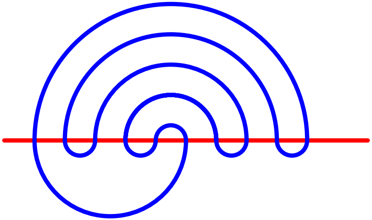
Masur-Veech Volumes of the Moduli Space of Quadratic Differentials

Quan Nguyen
Supervised by Paul Norbury

24th Oct 2025

Meanders



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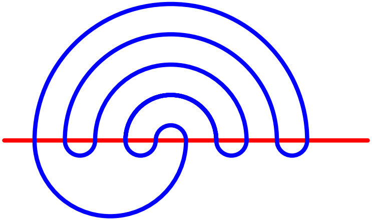

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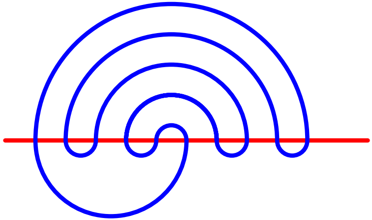

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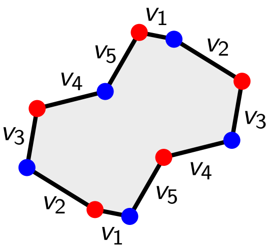
The asymptotics of $M(N)$ as $N \to \infty$ remains conjectural.

Translation surfaces/Abelian differentials



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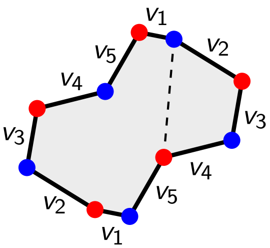

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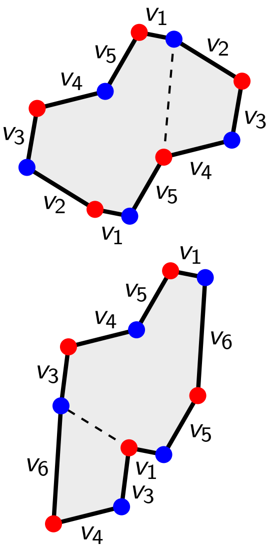

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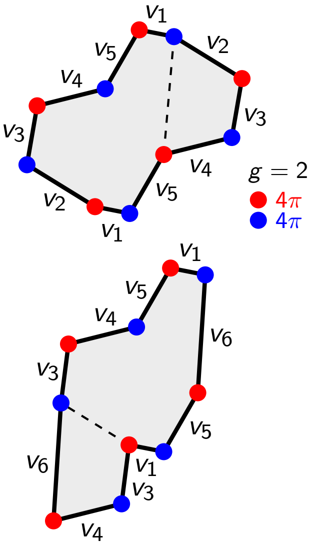

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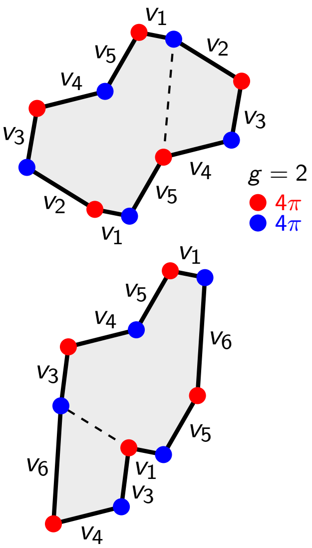

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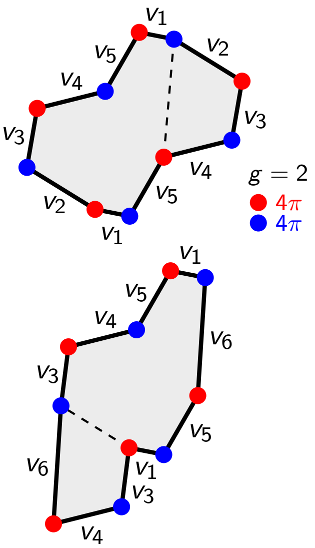

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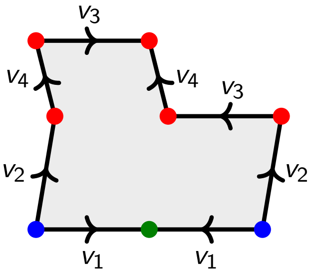
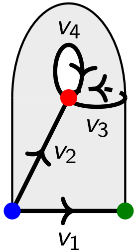
Cone points with angle $2\pi(k+1) \leftrightarrow$ order k singularities of ω.

Riemann-Roch: $\sum_i k_i = 2g - 2$.

Half-translation surface/quadratic differentials





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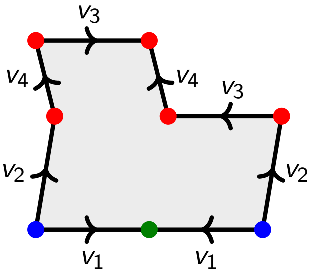

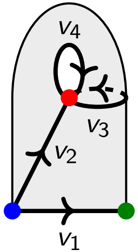

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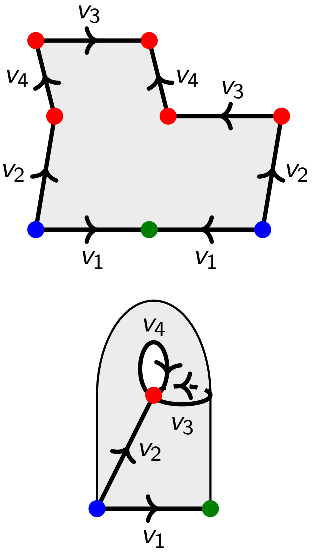

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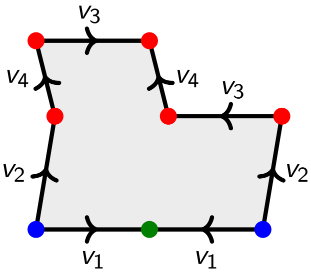

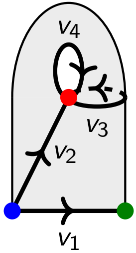

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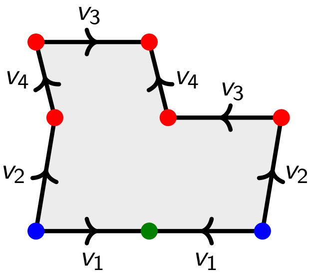
Simple poles $\leftrightarrow$ cone angles of π.

Riemann-Roch: $\sum_i k_i = 4g - 4$.

Moduli space and strata



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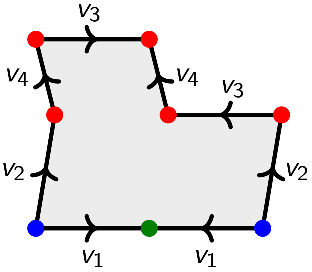

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Stratified by singularity orders. $\mathcal{H}(\mu)$ resp. $\mathcal{Q}(\mu)$ are "subsets" with singularity orders encoded by a partition μ.

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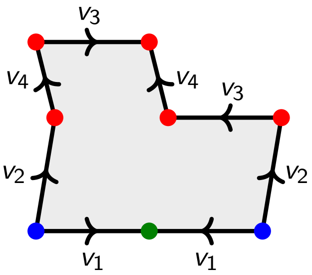

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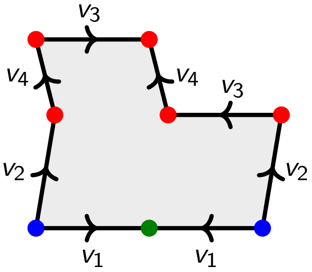

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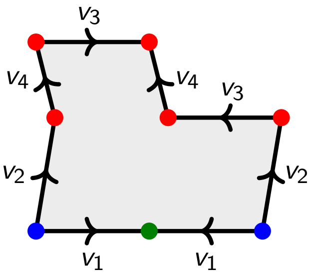

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Locally modelled on $\mathbb{C}^d$ - edges define coordinates, hence there is a natural volume.

The Masur-Veech volume

Theorem (Masur, Veech)

The space of flat surfaces with fixed finite area has finite volume.

$$\mathcal{H}_1(\mu) = \left\{ (X,\omega) \in \mathcal{H}(\mu) : \frac{i}{2}\int_X \omega \wedge \overline{\omega} = 1 \right\}$$

$$\mathcal{Q}_1(\mu) = \left\{ (X,q) \in \mathcal{Q}(\mu) : \int_X |q| = 1/2 \right\}$$

It can be shown that

$$\operatorname{Vol} \mathcal{Q}_1(\mu) = 2 \dim_{\mathbb{C}} \mathcal{Q}(\mu) \cdot \lim_{N\to\infty} \frac{1}{N^d} \cdot \underbrace{\#\mathcal{ST}(\mathcal{Q}(\mu), 2N)}_{\substack{\text{number of square-tiled} \\ \text{surfaces built from at} \\ \text{most } 2N \text{ squares}}} .$$

Volume computation preview

To compute volumes, we will:

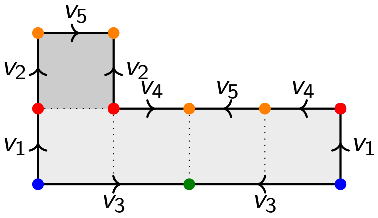
- Associate polynomials P_Γ to different volume contributions
- Apply an operator $\mathcal{Z}$ to evaluate volumes
- Sum all volume contributions

$$\mathcal{Z}(b_1^2 b_2^5) = 2! \cdot 5! \cdot \zeta(3) \cdot \zeta(6).$$

Goal: Compute $\#\mathcal{ST}(\mathcal{Q}(\mu), 2N)$

Cylinder decomposition

Every square-tiled surface has a unique horizontal cylinder decomposition. Consider the following square-tiled surface in $\mathcal{Q}(-1^2, 1^2)$.



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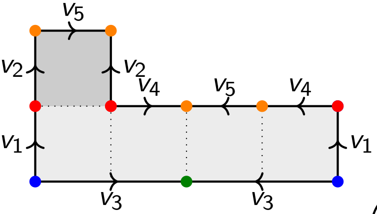
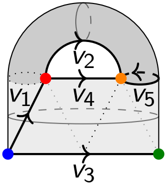
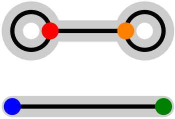
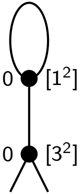






$\in \mathcal{R}_{0,3}^{[3^2]}$

$\in \mathcal{R}_{0,1}^{[1^2]}$



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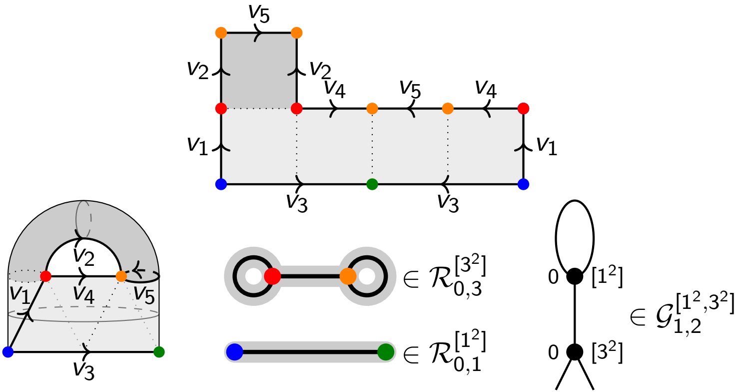
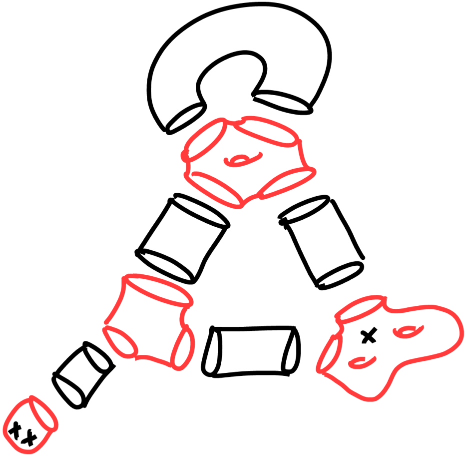
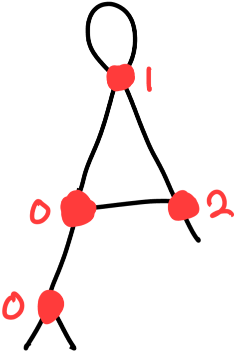


Note: Order k singularities correspond to $(k+2)$-valent vertices.

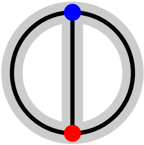
Cylinder Decomposition

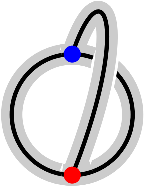




Ribbon graphs

Ribbon graphs are graphs endowed with a cyclic ordering of half-edges at the vertices.

$G_\Theta =$ 

$G_{\tilde{\Theta}} =$ 

They define an embedding of the underlying graph onto an oriented surface.

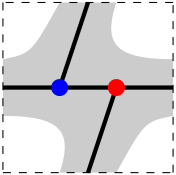


Figure: Embedding of $G_{\tilde{\Theta}}$ into a torus with 1 boundary

Metric ribbon graphs

We count metrics on ribbon graphs with fixed boundary lengths. Let A_G be the incidence matrix of a ribbon graph G. The space of metrics is the convex polytope

$$P_G(\boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}_+^{E(G)} : \boldsymbol{b} = A_G \boldsymbol{x}\}.$$

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The number of integer metrics is

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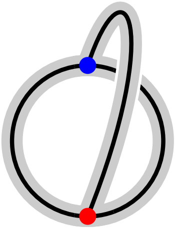
valence κ is

$$\mathcal{N}_{g,n}^{\kappa}(\boldsymbol{b}) = \sum_{G \in \mathcal{R}_{g,n}^{\kappa}} \frac{\mathcal{N}_G(\boldsymbol{b})}{|\mathrm{Aut}(G)|}.$$

Example of metric ribbon graphs



The incidence matrix is $\begin{pmatrix} 2 & 2 & 2 \end{pmatrix}$. The number of integer metrics correspond to positive integer solutions to $x_1 + x_2 + x_3 = b/2$.

$$\mathcal{N}_G(b_1) = \binom{b_1/2 - 1}{2} = \frac{(b_1 - 4)(b_1 - 2)}{8}.$$

Counting square-tiled surfaces of type Γ

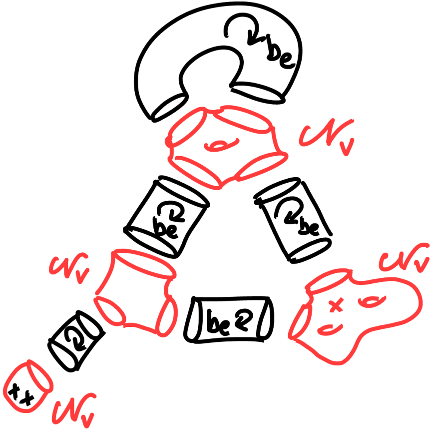
Lemma: If κ has at least one odd component, the total number of square-tiled surfaces in $\mathcal{Q}(\mu)$ of type $\Gamma \in \mathcal{G}_{g,n}^{\kappa}$ constructed from at most $2N$ squares of size $(1/2 \times 1/2)$ is

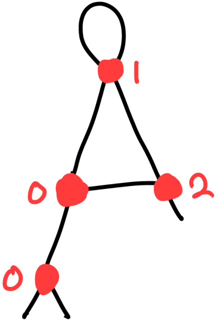
$$\#\mathcal{ST}_{\Gamma}(\mathcal{Q}(\mu), 2N) = \frac{2^d \cdot c_{\kappa}}{|\mathrm{Aut}(\Gamma)|} \sum_{\substack{\boldsymbol{b},\boldsymbol{H} \in \mathbb{N}^{E(\Gamma)} \\ \boldsymbol{b}\cdot\boldsymbol{H} \leq N \\ \boldsymbol{b} \in \mathbb{L}_{\Gamma}}} \prod_{e \in E(\Gamma)} b_e \cdot \prod_{v \in V(\Gamma)} \mathcal{N}_v(\boldsymbol{b})$$

where $\mathcal{N}_v(\boldsymbol{b})$ counts the number of ways cylinders of width b_i can be glued at vertex v. In particular, it is

$$\mathcal{N}_v(\boldsymbol{b}) = \mathcal{N}_{g_v,n_v}^{\kappa_v}((b_e)_{e \in E_v(\Gamma)})$$

Proof





Special case

Lemma

Suppose the top degree of the summand is a polynomial. Define

$$P_\Gamma = \frac{1}{2^{\#V(\Gamma)-1}} \cdot \frac{1}{|\mathrm{Aut}(\Gamma)|} \cdot \prod_{e \in E(\Gamma)} b_e \cdot \prod_{v \in V(\Gamma)} \mathcal{N}_v(\boldsymbol{b}).$$

The volume contribution of type Γ *is given by*

$$\mathrm{Vol}(\Gamma) = C_\kappa \cdot \mathcal{Z}(P_\Gamma)$$

where $\mathcal{Z}$ *is defined by*

$$\mathcal{Z} : \prod_{i=1}^{k} b_i^{m_i} \mapsto \prod_{i=1}^{k} m_i! \cdot \zeta(m_i + 1).$$

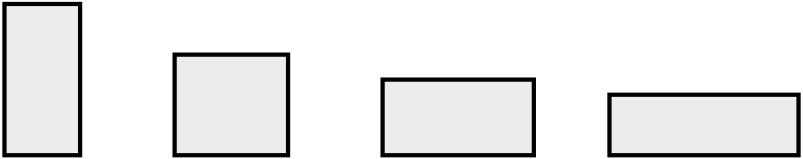
Why does the zeta function appear?

A small case:

$$\sum_{\substack{b,H\in\mathbb{N}\\ bH\leq N}} b^m = \sum_{H\in\mathbb{N}} \sum_{\substack{b\in\mathbb{N}\\ b\leq N/H}} b^m$$

$$\approx \sum_{H\in\mathbb{N}} \int_0^{N/H} b^m db$$

$$= \frac{N^{m+1}}{m+1} \sum_{H\in\mathbb{N}} \frac{1}{H^{m+1}}$$



Extention to a piecewise polynomial

Lemma

Suppose P_Γ is a piecewise polynomial of the form

$$b_1^{m_1} \cdots b_k^{m_k} \mathbb{1}_{\{b_1 = 2b_2\}}$$

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$$\mathcal{Z}\left(\prod_{i=1}^{k} b_i^{m_i} \mathbb{1}_{\{b_1 = 2b_2\}}\right) = (m_1 + m_2)! \cdot \xi(m_1, m_2) \prod_{i \geq 3} m_i! \cdot \zeta(m_i + 1),$$

$$\xi(m_1, m_2) := 2^{m_1} \zeta(m_1 + m_2) - (2^{m_1} + 2^{-m_2 - 1}) \zeta(m_1 + m_2 + 1).$$

Example: $\mathcal{Q}(-1^2, 2)$

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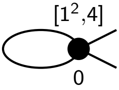
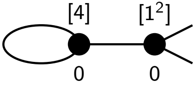
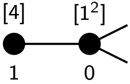
$\mathcal{G}_{1,2}^{[1^2,4]}$	$P_\Gamma \mapsto C_{[1^2,4]} \cdot \mathcal{Z}(P_\Gamma)$
	
	
	

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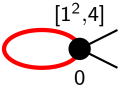
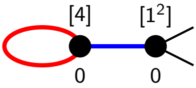
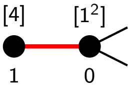
	$\frac{1}{2^0} \cdot \frac{1}{2} \cdot b_1 \cdot \mathcal{N}_{0,2}^{[1^2,4]}(b_1, b_1)$ $= \frac{b_1^2}{2}$
	$\frac{1}{2^1} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot \mathcal{N}_{0,3}^{[4]}(b_1, b_1, b_2) \cdot \mathcal{N}_{0,1}^{[1^2]}(b_2)$ $= \frac{b_1 b_2}{4} \cdot \mathbb{1}_{\{b_2 = 2b_1\}}$
	$\frac{1}{2^1} \cdot 1 \cdot b_1 \cdot \mathcal{N}_{1,1}^{[4]}(b_1) \cdot \mathcal{N}_{0,1}^{[1^2]}(b_1)$ $= \frac{b_1^2}{16}$

Example: $\mathcal{Q}(-1^2, 2)$

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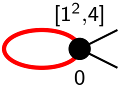
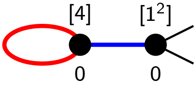
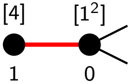
	$\frac{1}{2^0} \cdot \frac{1}{2} \cdot b_1 \cdot \mathcal{N}_{0,2}^{[1^2,4]}(b_1, b_1)$ $= \frac{b_1^2}{2} \mapsto 8\zeta(3)$
	$\frac{1}{2^1} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot \mathcal{N}_{0,3}^{[4]}(b_1, b_1, b_2) \cdot \mathcal{N}_{0,1}^{[1^2]}(b_2)$ $= \frac{b_1 b_2}{4} \cdot \mathbb{1}_{\{b_2 = 2b_1\}} \mapsto 8\zeta(2) - 9\zeta(3)$
	$\frac{1}{2^1} \cdot 1 \cdot b_1 \cdot \mathcal{N}_{1,1}^{[4]}(b_1) \cdot \mathcal{N}_{0,1}^{[1^2]}(b_1)$ $= \frac{b_1^2}{16} \mapsto \zeta(3)$

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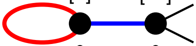
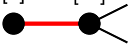
$[4]$ $[1^2]$  0 0	$\frac{1}{2^1} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot \mathcal{N}_{0,3}^{[4]}(b_1, b_1, b_2) \cdot \mathcal{N}_{0,1}^{[1^2]}(b_2)$ $= \frac{b_1 b_2}{4} \cdot \mathbb{1}_{\{b_2 = 2b_1\}} \mapsto 8\zeta(2) - 9\zeta(3)$
$[4]$ $[1^2]$  1 0	$\frac{1}{2^1} \cdot 1 \cdot b_1 \cdot \mathcal{N}_{1,1}^{[4]}(b_1) \cdot \mathcal{N}_{0,1}^{[1^2]}(b_1)$ $= \frac{b_1^2}{16} \mapsto \zeta(3)$

$$\operatorname{Vol} \mathcal{Q}(-1^2, 2) = 8\zeta(3) + 8\zeta(2) - 9\zeta(3) + \zeta(3) = \frac{4\pi^2}{3}.$$

Volumes of principal strata

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There is a deep connection between counts of trivalent ribbon graphs (which can be extended to include univalent vertices) and intersection theory on the moduli space of curves due to Kontsevich.

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The tautological line bundles $\mathcal{L}_i$ are given by $T^*_{p_i}X$ at $X \in \overline{\mathcal{M}}_{g,n}$.

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In 1992, Kontsevich proved this conjecture... using ribbon graphs!

Counts of trivalent ribbon graphs

Kontsevich proves the *Main Identity*:

$$\sum_{\substack{\boldsymbol{d}\in\mathbb{N}_0^n \\ |\boldsymbol{d}|=3g-3+n}} \frac{\langle \tau_{d_1}\cdots\tau_{d_n}\rangle}{2^{5g-5+2n}} \prod_{i=1}^{n} \frac{(2d_i)!}{\lambda_i^{2d_i+1}} = \sum_{G\in\mathcal{R}_{g,n}^{\mathrm{tri}}} \frac{1}{|\mathrm{Aut}(G)|} \prod_{e\in E(G)} \frac{1}{\widetilde{\lambda}(e)}$$

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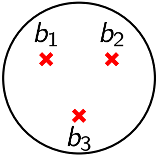
Using Norbury's lattice point counts, we obtain:

$$N_{g,n}(b_1,\ldots,b_n) = \mathcal{N}_{g,n}(b_1,\ldots,b_n) + \text{lower order terms.}$$

The combinatorial moduli space $\mathcal{M}_{g,n}^{\mathrm{comb}}$



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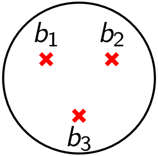

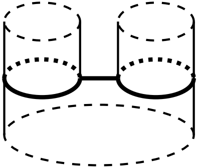

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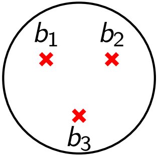

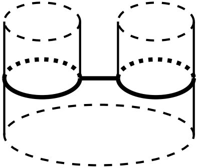

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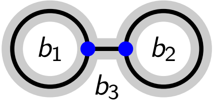

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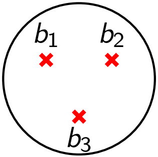
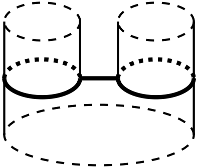


$$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \quad = \quad \{\text{decorated curves}\}$$



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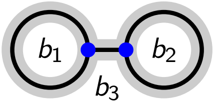

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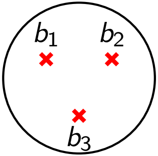

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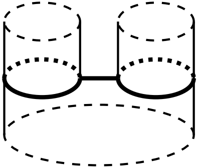

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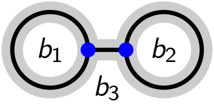

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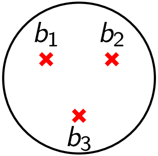

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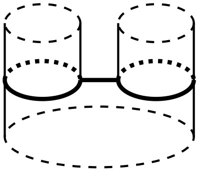

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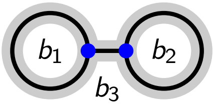

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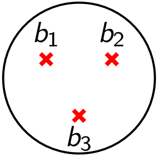

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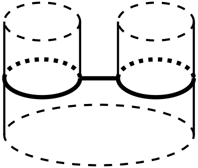

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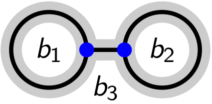

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Integrating the top form $\exp \Omega = \frac{\Omega^{3g-3+n}}{(3g-3+n)!}$ gives the symplectic volume:

$$V_{g,n}^S(\boldsymbol{b}) = \int_{\overline{\mathcal{M}}_{g,n}(\boldsymbol{b})} \exp \Omega = \sum_{\substack{\boldsymbol{d} \in \mathbb{N}_0^n \\ |\boldsymbol{d}| = 3g-3+n}} \frac{\langle \tau_{d_1} \cdots \tau_{d_n} \rangle}{d_1! \cdots d_n!} b_1^{d_1} \cdots b_n^{d_n}.$$

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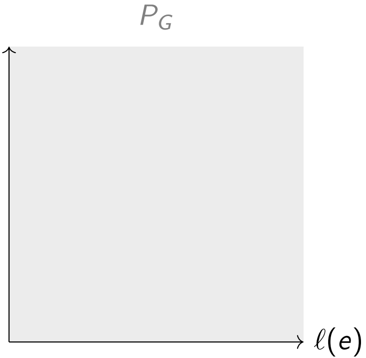

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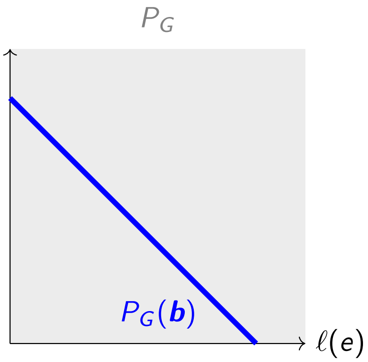

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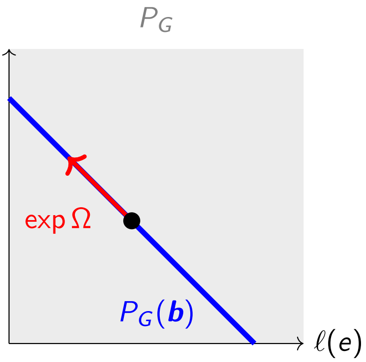

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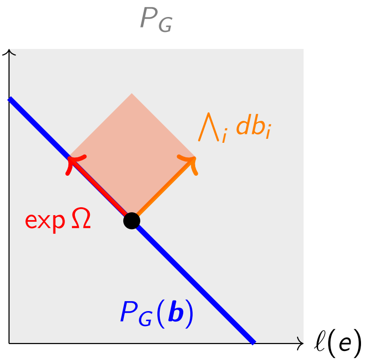

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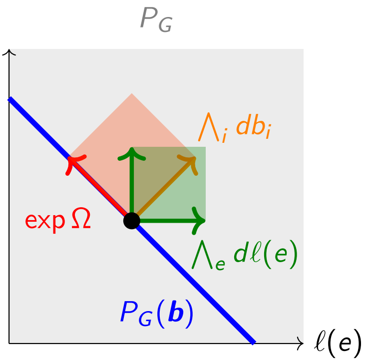

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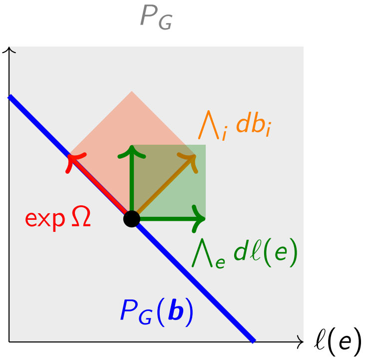

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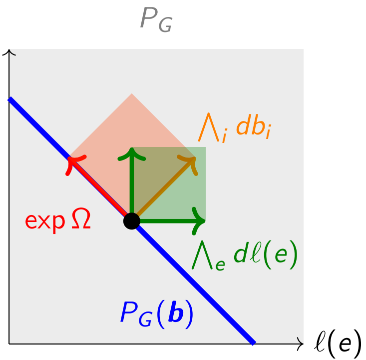
Kontsevich proves the ratio of measures is $\rho = 2^{5g-5+2n}$, hence $V^E_{g,n}(\boldsymbol{b})$ is a constant multiple ρ of $V^S_{g,n}(\boldsymbol{b})$.

Kontsevich's proof (part 2)



The top form $\exp\Omega$ defines the symplectic fibre volume $V^S_{g,n}(\boldsymbol{b})$.

Taking the Laplace transform allows us to integrate the top form $\exp\Omega \bigwedge_i db_i$ in $\mathcal{M}^{\mathrm{comb}}_{g,n}$.

The top form $\bigwedge_e d\ell(e)$ defines the Euclidean volume $V^E_{g,n}(\boldsymbol{b})$.

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We now know how to count trivalent ribbon graphs!

Counting trivalent ribbon graphs with univalent vertices

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Theorem

Let $\mathcal{N}_{g,n,p}(b_1,\ldots,b_n)$ be the weighted count of trivalent integral metric ribbon graphs with p univalent vertices. Then

$$\mathcal{N}_{g,n,p}(b_1,\ldots,b_n) = N_{g,n+p}(b_1,\ldots,b_n,\underbrace{0,\ldots,0}_{p}) + \text{lower order terms.}$$

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This uses the string equation

$$\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n+1} = \sum_{i=1}^{n} \langle \tau_{d_1} \cdots \tau_{d_i - 1} \cdots \tau_{d_n} \rangle_{g,n}$$

and Kontsevich's Main Identity.

Volume formula for the principal strata

Theorem

The volume of the principal stratum $\mathcal{Q}(-1^n, 1^{4g-4+n})$ is

$$\operatorname{Vol} \mathcal{Q}(-1^n, 1^{4g-4+n}) = C_{g,n} \cdot \sum_{\Gamma \in \mathcal{G}_{g,n}} \mathcal{Z}(P_\Gamma)$$

where

$$P_\Gamma = \frac{1}{2^{\# V(\Gamma)-1}} \cdot \frac{1}{|\mathrm{Aut}(\Gamma)|} \cdot \prod_{e \in E(\Gamma)} b_e \cdot \prod_{v \in V(\Gamma)} N_{g_v, n_v + p_v}\left((b_e)_{e \in E_v(\Gamma)}, 0^{p_v}\right)$$

and

$$C_{g,n} = \frac{2^{5g-6+2n}(4g-4+n)!}{(6g-7+2n)!}$$

Example: Volume of $\mathcal{Q}(-1, 1^5)$

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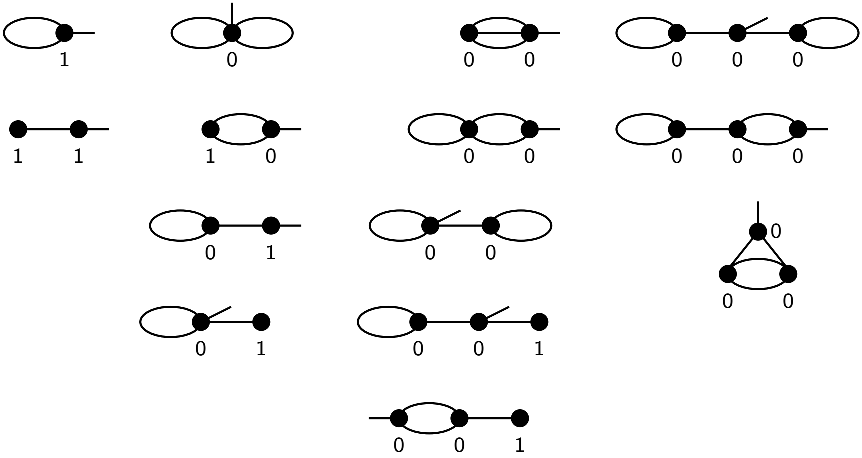
Stable graphs in $\mathcal{G}_{2,1}$:



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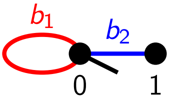
$\Gamma =$ 

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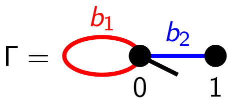

$$P_\Gamma = \frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, 0) N_{1,1}(b_2)$$

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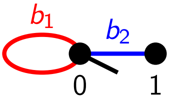

$$P_\Gamma = \frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, 0) N_{1,1}(b_2) = \frac{1}{384} \cdot b_1^3 b_2^3 + \frac{1}{768} \cdot b_1 b_2^5$$

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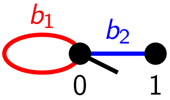
$\Gamma =$ 

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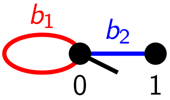
$\Gamma =$ 

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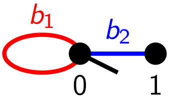
$\Gamma =$ 

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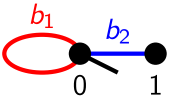
$\Gamma =$ 

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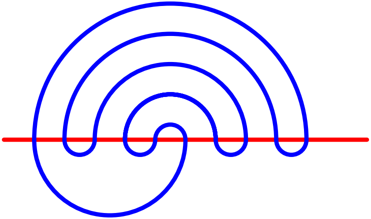
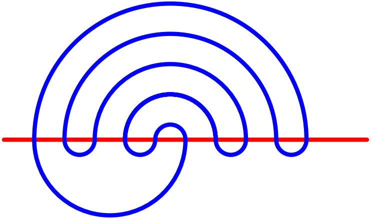
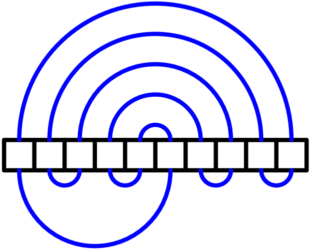
Meanders



Meanders





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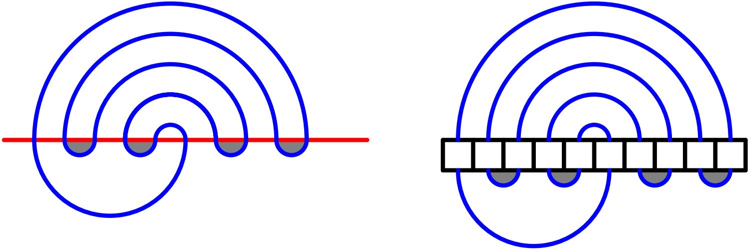
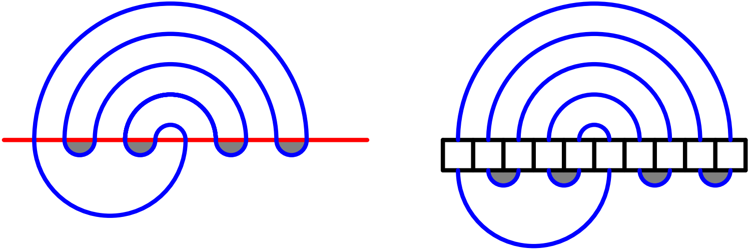


Counting meanders with n minimal arcs corresponds to counting square-tiled surfaces in $\mathcal{Q}_{0,n}$ with one horizontal and one vertical cylinder. Its volume contribution corresponds to the asymptotics



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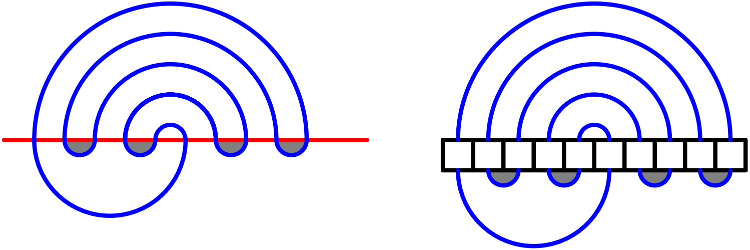
The distribution of single horizontal and vertical cylinders are asymptotically independent.

$$\frac{\mathrm{Cyl}_{1,1}\, \mathcal{Q}(-1^{n+1}, 1^{n-3})}{\mathrm{Cyl}_1\, \mathcal{Q}(-1^{n+1}, 1^{n-3})} = \frac{\mathrm{Cyl}_1\, \mathcal{Q}(-1^{n+1}, 1^{n-3})}{\mathrm{Vol}\, \mathcal{Q}(-1^{n+1}, 1^{n-3})}$$



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$$\mathcal{M}_n(N) = \frac{2(n+1)\,\mathrm{Cyl}_{1,1}\,\mathcal{Q}(-1^{n+1}, 1^{n-3})}{(n+1)!(n-3)!(4n-8)} \cdot N^{2n-4} + o(N^{2n-4})$$

The end

Thankyou!